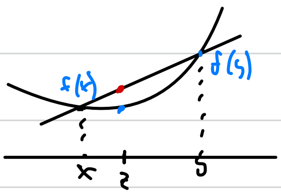
Math 564: Advance Analysis 1
Lecture 25

Recall. A function $f:(a,b)\to\mathbb{R}$ is called convex (resp. concave) if $\forall x,y\in(a,b)$, $\forall \alpha\in[0,1]$, we have $f(\alpha\cdot x+(1-\alpha)y)\le \alpha f(x)+(1-\alpha)\cdot f(y)$ (resp. $\ge$).



convex f

Prop. f is convex $\iff \dfrac{f(x_2)-f(x_1)}{x_2-x_1}$ is increasing in x_1.

$$[\text{If } f'' \text{ exists}] \iff f''\ge 0.$$

Examples. (a) For $\alpha>0$, $t\mapsto t^\alpha$ is convex $\iff \alpha\ge 1$, and concave $\iff \alpha<1$.
(b) $t\mapsto e^t$ is convex, while $t\mapsto \log t$ is concave.

Minkowski's inequality (Δ-ineq. for L^p, $p\ge 1$). $\|f+g\|_p\le \|f\|_p+\|g\|_p$ for all $p\ge 1$.

Proof. If f or g are 0, this is trivial, so suppose $\|f\|_p,\|g\|_p>0$. Normalizing by $\|f\|_p+\|g\|_p$ (dividing both sides), we may assume WLOG that $\|f\|_p+\|g\|_p=1$. Under this assumption, we need to prove $\|f+g\|_p\le 1$. By triangle ineq. $|f+g|\le |f|+|g|$, so $|f+g|^p\le(|f|+|g|)^p$, hence it is enough to prove $\|f+g\|_p\le 1$ for $f,g\ge 0$.
Raising to the power of p, we need to prove $\int (f+g)^p\,d\mu\le 1$.
Letting $\alpha:=\|f\|_p$, so $\|g\|_p=(1-\alpha)$, we write $f=\alpha\cdot F$ and $g=(1-\alpha)\cdot G$, for some norm 1 functions F, G, namely $F:=\frac{1}{\|f\|_p}\cdot f$ and $G:=\frac{1}{\|g\|_p}\cdot g$.
Now convexity of $(\cdot)^p$ applies:
$(f+g)^p=(\alpha\cdot F+(1-\alpha)\cdot G)^p\le \alpha\cdot F^p+(1-\alpha)G^p$, integrating which, we get:

$$\int (f+g)^p\le \alpha\cdot\int F^p+(1-\alpha)\int G^p=\alpha\cdot\|F\|_p^p+(1-\alpha)\cdot\|G\|_p^p=\alpha\cdot 1+(1-\alpha)\cdot 1=1.$$

$\square$

Thus, L^p spaces, $1 \le p < \infty$, are normed vector spaces. In fact:

Thm. For $1 \le p < \infty$, L^p is a Banach space.

Proof. Recall that we need to show that if a series $\sum_n f_n$ converges absolutely, i.e. $\sum_n \|f_n\|_p < \infty$, then it converges in norm.
By Minkowski's inequality, we have $\| \sum_{i \le n} |f_i| \|_p \le \sum_{i \le n} \|f_i\|_p \le \sum_n \|f_n\|_p < \infty$.

Thus, $g := \sum_{n \in \mathbb{N}} |f_n| \in L^p$ hence $\|g\|_p = \| \sum_n |f_n| \|_p \le \sum_n \|f_n\|_p < \infty$.

In particular, $g < \infty$ a.e. hence the series $\sum_n f_n(x)$ converges for a.e. x to a limit $f(x)$.
Note that $|f - \sum_{i \le n} f_i|^p \le \left(|f| + \sum_{i \le n} |f_i|\right)^p \le (2g)^p = 2^p \cdot g^p \in L^1$,

so by the DCT, $\int |f - \sum_{i \le n} f_i|^p \, d\mu \to 0$, hence $\|f - \sum_{i \le n} f_i\|_p \to 0$. $\square$

Thm. The set of simple functions is dense in $L^p(X, \mu)$.

Proof. By writing $f = f_+ - f_-$, it is enough to prove for $f \ge 0$.
Let $f_n \to f$ be an increasing sequence of non-neg. simple functions, so $f_n^p \to f^p$ and $(f - f_n)^p \le 2^p f^p$, so by the DCT, $\|f - f_n\|_p \to 0$. $\square$

Cor. If (X, μ) is ctbly generated (mod μ-null), then $L^p(X, \mu)$ is separable, hence Polish, for $1 \le p < \infty$.

L^∞ space. For (X, μ) measure space and $f : X \to \mathbb{R}$ μ-measurable.
We'd like to define $\|f\|_\infty$ as the $\sup_{x \in X} |f(x)|$ but this would depend on the representative f of the almost equality class of f. Note that $\sup_{x \in X} |f(x)| = \inf \{c \ge 0 : |f| \le c\}$.
This last version can be modified to be invariant under null sets.

$$\|f\|_\infty := \inf\{c \geq 0 : |f| \leq c \text{ a.e.}\}.$$

In fact, because ctbl unions of null is null, this inf = min.

$\|f\|_\infty$ is called the essential supremum of $|f|$.

Let $L^\infty(X,\mu)$ be the set (mod null) of μ-measurable functions f with $\|f\|_\infty < \infty$.

Obs. If $f \in L^\infty(X,\mu)$, then $\exists\, \tilde{f} = f$ a.e. and $\|\tilde{f}\|_\infty = \sup_{x\in X}|\tilde{f}(x)| = \|f\|_\infty$.
In other words, we can always work with representatives of almost equality classes, where $\|f\|_\infty$ is an honest supremum.

In other words, WLOG, we can think of $L^\infty(X,\mu)$ as the space (mod null) of bdd measurable functions with the usual sup-norm.

Prop. (a) $\|\cdot\|_\infty$ is a norm (obeys $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$) on $L^\infty(X,\mu)$.
(b) $L^\infty(X,\mu)$ is a Banach space.
(c) Simple functions are dense in $L^\infty(X,\mu)$.

However, $L^\infty(X,\mu)$ is almost never separable (unless X is finite and μ is the counting measure, like $\mathbb{R}^d$).

Def. Let A be a (discrete) set with μ the counting measure. We denote by $\ell^p(A) := L^p(A,\mu)$, for $1 \leq p \leq \infty$. In particular, $\ell^p(d) = \mathbb{R}^d$, where $d = \{0,1,\dots,d-1\}$.

Prop. $\ell^\infty(\mathbb{N})$ is not separable.

Proof. Note that $\ell^\infty(\mathbb{N}) = \bigcup_{n\in\mathbb{N}} n^{\mathbb{N}}$ = the set of bdd sequences.
In particular, the continuum $2^{\mathbb{N}} \subseteq \ell^\infty(\mathbb{N})$ is discrete subset, indeed $\forall x,y \in 2^{\mathbb{N}}$ distinct, $\|x-y\|_\infty = \sup_n |x(n)-y(n)| = 1$. $\square$

Algebraic properties of L^p spaces. We want to understand if $f \in L^p$ and $g \in L^q$, then $f \cdot g \in L^?$.

Prop. Algebraic average $\geq$ geometric average, i.e. $\forall\, a, b > 0$ and $\alpha \in (0,1)$, we have

$$\alpha \cdot a + (1-\alpha) b \geq a^{\alpha} \cdot b^{(1-\alpha)}.$$

Proof. This is just the convexity of $t \mapsto e^t$ because writing $a = e^A$, $b = e^B$, so $\alpha \cdot a + (1-\alpha) b = \alpha \cdot e^A + (1-\alpha) \cdot e^B \geq e^{\alpha A + (1-\alpha) B} = (e^A)^{\alpha} \cdot (e^B)^{(1-\alpha)} =$
$= a^{\alpha} \cdot b^{(1-\alpha)}$. $\square$

Hölder's inequality. If $f \in L^p$ and $g \in L^q$ then $f \cdot g \in L^r$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In fact:

$$\|f \cdot g\|_r \leq \|f\|_p \cdot \|g\|_q.$$

In particular, $\|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q$ if p and q are conjugate exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By replacing f and g with f^r and g^r and p, q with $\frac{p}{r}$ and $\frac{q}{r}$, reduces to proving for $r = 1$. We may assume $\|f\|_p, \|g\|_q > 0$. Dividing both sides by $\|f\|_p \cdot \|g\|_q$, allows us to assume that $\|f\|_p = 1$, $\|g\|_q = 1$. So we need to show

$$\|f \cdot g\|_1 \leq 1.$$

$$|f \cdot g| = |f| \cdot |g| = (|f|^p)^{\frac{1}{p}} \cdot (|g|^q)^{\frac{1}{q}} \overset{\text{prop. above}}{\leq} \frac{1}{p} \cdot |f|^p + \frac{1}{q} \cdot |g|^q,$$ so integrating:

$$\int |fg| \, d\mu \leq \frac{1}{p} \int |f|^p \, d\mu + \frac{1}{q} \int |g|^q \, d\mu = \frac{1}{p} \cdot \|f\|_p^p + \frac{1}{q} \cdot \|g\|_q^q = \frac{1}{p} + \frac{1}{q} = 1.$$ $\square$